EVLM

Numerical integration

Problem formulation

Let $y = f(x)$ be a function, defined and Riemann-integrable in the interval $[a,b]$. Considering a given table of values $y_i = f(x_i)$ of the function in the points (nodes) $x_0, x_1, x_2, ..., x_n \in [a,b]$, find the approximate value of the integral $I = \int_a^b f(x)dx$.

Newton-Cotes quadrature formulas

If n is a natural number and $h = \frac{b-a}{n}$ is a step, with the help of which the interval of iteration is divided into n equal sub-intervals, which means that $x_0 = a$, $x_{i+1} = x_i + h$, $i = 0,1,...,n-1$, then the most frequently used sum quadrature formulas of Newton-Cotes are given in the table below. Here $M_k = \max\limits_{x\in[a,b]} \left|f^{(k)}(x)\right|$, under condition that there exists a continuous k-th derivative of $y = f(x)$.

Name	Quadrature formula for numerical integration	Error evaluation $\lvert R(f,x)\rvert$
Left rectangle formula	$I_1 \approx h\sum\limits_{i=0}^{n-1} f(x_i)$	$\frac{(b-a)^2}{2n}M_1$, i.e. $\frac{(b-a)M_1}{2}h$
Right rectangle formula	$I_2 \approx h\sum\limits_{i=1}^{n} f(x_i)$	$\frac{(b-a)^2}{2n}M_1$, i.e. $\frac{(b-a)M_1}{2}h$
Mid-rectangle formula	$I_3 \approx h\sum\limits_{i=0}^{n-1} f(x_i + \frac{h}{2})$	$\frac{(b-a)^3}{24n^2}M_2$, i.e. $\frac{(b-a)M_2}{24}h^2$
Trapezoid formula	$I_T \approx \frac{h}{2}\left(f(a) + 2\sum\limits_{i=1}^{n-1} f(x_i) + f(b)\right)$	$\frac{(b-a)^3}{12n^2}M_2$, i.e. $\frac{(b-a)M_2}{12}h^2$
Simpson's formula	$n = 2m$, $I_S \approx \frac{h}{3}\left(f(a) + 4\sum\limits_{i=1}^{m} f(x_{2i-1}) + 2\sum\limits_{i=1}^{m-1} f(x_{2i}) + f(b)\right)$	$\frac{(b-a)^5}{180n^4}M_4$, i.e. $\frac{(b-a)M_4}{180}h^4$

Example 1. The values for the function $y = f(x) = \ln(x^2)$ in the interval $[2,3]$ are given in the first two columns of table 1. Calculate the approximate values of the integral $I = \int_2^3 f(x)dx$, using all the quadrature formulas from the table above.

Solution:

In this case we have a step of $h = 0{,}1$ and the number of sub-intervals ise $n = 10$.

Table 1

i	x_i	y_i
0	2,0	1,38629
1	2,1	1,48387
2	2,2	1,57691
3	2,3	1,66582
4	2,4	1,75094
5	2,5	1,83258
6	2,6	1,91102
7	2,7	1,98650
8	2,8	2,05924
9	2,9	2,12942
10	3,0	2,19722

Table 2

i	$x_i + \frac{h}{2}$	$y_{i+\frac{1}{2}}$
0	2,05	1,43568
1	2,15	1,53094
2	2,25	1,62186
3	2,35	1,70883
4	2,45	1,79218
5	2,55	1,87219
6	2,65	1,94912
7	2,75	2,02320
8	2,85	2,09464
9	2,95	2,16361

a) Using the left rectangle formula we have to add together the values from y_0 to y_9 and to multiply the resulting number by h. We get

$$I_1 \approx h\sum_{i=0}^{n-1} f(x_i) = h\sum_{i=0}^{9} y_i = 0{,}1.(17{,}7826) = 1{,}77826\,.$$

b) Using the right rectangle formula we find the sum from y_1 to y_{10} and the resulting number we multiply by h. We get

$$I_2 \approx h\sum_{i=1}^{n} f(x_i) = h\sum_{i=1}^{10} y_i = 0{,}1.18{,}5935 = 1{,}85935\,.$$

c) In the case of the mid-rectangle formula using table 2 we calculate:

$$I_3 \approx h\sum_{i=0}^{n-1} f(x_i + \frac{h}{2}) = h\sum_{i=0}^{9} y_{i+\frac{1}{2}} = 0{,}1\ .\ 18{,}1923{=}1{,}81923.$$

d) Respectively for trapezoids we have

$$I_T \approx \frac{h}{2}\left(f(a) + 2\sum_{i=1}^{n-1} f(x_i) + f(b) \right) = \frac{h}{2}\left(y_0 + 2\sum_{i=1}^{9} y_i + y_{10} \right)$$
$$= 0{,}05(y_0 + 2(y_1 + y_2 + ... + y_9) + y_{10}) = 0{,}05(1{,}38629 + 2.(16{,}39631) + 2{,}19722) =$$
$$0{,}05.(36{,}37615) = 1{,}818807\,;$$

e) Using Simpson’s summation formula:

$$I_S \approx \frac{h}{3}\left(f(a) + 4\sum_{i=1}^{m} f(x_{2i-1}) + 2\sum_{i=1}^{m} f(x_{2i}) - f(b) \right)$$
$$= \frac{0,1}{3}(y_0 + 4(y_1 + y_3 + ... + y_9) + 2(y_2 + y_4 + ... + y_8) + y_{10})$$
$$= \frac{0,1}{3}\left(1,38629 + 4.(9,09820) + 2.(7,29812) + 2,19722\right) = \frac{0,1}{3}.(54,57254) = 1,819085 .$$

Example 2. Evaluate the error of numerical integration for $I_1, I_{T,} I_S$ from example 1.

Solution:

We find one by one the derivatives of $f(x) = \ln(x^2)$:

$$f'(x) = \frac{2}{x},\quad f''(x) = \frac{-2}{x^2},\quad f'''(x) = \frac{4}{x^3},\quad f^{(4)}(x) = \frac{-12}{x^4}.$$

Then $M_1 = \max\limits_{2\le x\le 3} |f'(x)| = f'(2) = 1$, $M_2 = \max\limits_{2\le x\le 3} |f''(x)| = |f''(2)| = \frac{1}{2}$,

$M_4 = \max\limits_{2\le x\le 3} \left|f^{(4)}(x)\right| = \left|f^{(4)}(2)\right| = \frac{3}{4}$.

By substituting in the error formula of the left rectangle method we get: $|R_1(f,h)| \le \frac{M_1(b-a)h}{2} = \frac{0,1}{2} = 0,05$ i.e. the first-second digit of the result is correct. Consequently after rounding off up to the second digit we have $I_1 \approx 1,78$.

For the trapezoid formula we calculate: $|R_T(f,h)| \le \frac{M_2(b-a)h^2}{12} = \frac{0,01}{24} \le 0,005$ or two-three digits after the decimal comma. Then $I_T \approx 1,819$.

For Simpson's quadrature formula:

$|R_S(f,h)| \le \frac{M_4(b-a)h^4}{180}$, i.e. all digits are correct and $I_S = \int_2^3 \ln(x^2)dx \approx 1,819085$.

Note. The formula of the function is not usually given or finding the derivative is difficult or there might not even exist derivatives for a given row. Then simpler formulas are used which give a small error. Accuracy in such cases is achieved by reducing the step h.

Example 3. Determine the step of numerical integration h so that the approximation of calculation of the integral $\int_{-1}^{3} \sqrt{1+x^2}\, dx$ using the trapezoid method guarantees accuracy of the result $\varepsilon = 0,000001$.

Solution:

Here $a = -1,\ b = 3$. For the derivatives we have: $f'(x) = \dfrac{x}{\sqrt{1+x^2}}$, $f''(x) = \dfrac{1}{(1+x^2)\sqrt{1+x^2}}$.

It is obvious that $0 \le f''(x) \le 1$ for every x, and consequently $M_2 \le 1$. By substituting in the error formula for the trapezoid method we find that:

$$|R_T(f,h)| \le \frac{M_2(b-a)h^2}{12} = \frac{(3-(-1))}{12}h^2 = \frac{h^2}{3}.$$

To guarantee the sought accuracy we introduce the condition $\dfrac{h^2}{3} \le \varepsilon$, or $h \le \sqrt{3\varepsilon} = \sqrt{0,000003} \approx 0,00173$. It is convenient to use the step $h = 0,001$ and accordingly to divide the interval $[-1,3]$ by $n = 4000$ equal sub-intervals. To carry out the calculations it is obvious that a computer is needed as well as a mediating accuracy of 10^{-8}.

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